PERGAMON

International Journal of Solids and Structures 37 (2000) 1975–1996

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Bifurcation results for plasticity coupled to damage with MCR-effect

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Received 24 April 1997; in revised form 20 November 1998

Abstract

The thermodynamic framework is outlined for a class of models, that involve the kinetic coupling of damage to inelastic deformation. This class includes effects of microcrack–closure–reopening (MCR). The general conditions for discontinuous bifurcations, which define the onset of band-shaped localization, are given for the considered class of models. Solutions for the band direction and the corresponding thermodynamic state are given for isotropic elastoplastic response at plane stress, whereby two different damage models that account for the MCR-effect in tension-compression, are employed. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

A vast amount of work has been devoted to establishing the necessary conditions for the existence of bifurcations of the incremental solution in solids involving rate-independent inelasticity, whereby the secondary solution is characterized by (weak or strong) discontinuities along a characteristic surface. One major reason for the interest in such (discontinuous) bifurcations is that they are frequently considered as precursors of localized failure phenomena (such as shear bands) which preceed fracture (cracks). At this junction we note that the traditional view on localization, is that a band (of unknown width) is trapped between two parallel characteristic surfaces, across which a jump in the rate of displacement gradient (or deformation gradient in the case of finite strains) is developing. Such a 'weak' discontinuity was assumed already by Hill (1962). In recent years, it has been shown that the same conditions apply for the emergence of a 'strong' discontinuity, which notion refers to an emerging discontinuity in the displacement rate itself (and not only its gradient), cf Steinmann et al. (1997).

Explicit bifurcation results (including the critical direction of the characteristic surface and the

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PII: S0020-7683(98)00344-8

corresponding value of the hardening/softening modulus) have been obtained in the literature for a variety of elastic–plastic models. Such predictions range from the classic works by Rudnicki and Rice (1975), Rice (1976) to the more recent contributions (generalizations) by Ottosen and Runesson (1991), Runesson et al. (1991), Bigoni and Hueckel (1991) and Neilsen and Schreyer (1993). The theoretical developments have been pursued in parallel with (and inspired by) achievements in the development of finite element algorithms for capturing localization phenomena, among which we may mention those of Ortiz et al. (1987), De Borst and Sluys (1991), Larsson et al. (1993), Simo et al. (1993), and Larsson and Runesson (1996).

By rephrasing elastic-damage models in a plasticity-like format, it is possible to use the results of, say, Ottosen and Runesson (1991), to obtain bifurcation results, as shown by Rizzi et al. (1995, 1996). Moreover, by coupling the development of damage kinetically to plastic deformation, cf Lemaitre (1992) and references therein, it is possible to obtain closed form solutions for bandshaped localization (as shown in this paper). Such results, although only for isotropic damage and restricting to a specific model a priori, were presented by Doghri and Billardon (1995).

When the development of damage is isotropic, it induces a scalar reduction of the (elastic part of) the free energy. However, it is well-known that such a simple model is not physically realistic. The need to model microcrack–closure–reopening (MCR) effects requires the incorporation of stress-induced anisotropy into the constitutive relations. A large number of such models have been proposed since the concept of 'maximum principal stress damage' was introduced by Hayhurst and Leckie (1973). Other contributions are those of Leckie and Onat (1981), Sidoroff (1981), Betten (1982), Ortiz (1985), Chow and Wang (1987), Murakami (1987), Yazdani and Schreyer (1988), Ju (1990), Carol et al. (1994) to mention a few. A recent review by Carol and Willam (1996) of a quite general class of models, comprising many of those mentioned above, has shown that most existing models are not thermodynamically consistent, i.e. the stress is not derivable from a free energy. Hence, energy can be dissipated or (even worse) produced within the 'elastic' range (when the damage variables are held constant). A thermodynamically consistent model is conveniently contained within the framework of 'generalized standard dissipative materials', originally proposed by Halphen and Son (1975), if certain non-standard features are allowed. Such non-standard extension refers to the necessity of invoking nonassociative dissipation rules w.r.t. the formulation of feasible damage rule(s).

The first purpose of this paper is to present a framework for models of plasticity coupled to damage that include MCR effects. The second purpose of the paper is to carry out a bifurcation analysis for the chosen class of models.

The paper is outlined as follows: a formulation of plasticity with kinetic coupling to damage is presented in Section 2, whereby the state of damage is defined by a single (scalar) variable. The 'equivalent strain' and 'effective stress' concepts are used, which means that virgin elastic properties define the effective stress for a given state of (elastic) strains, and this stress is used in the yield criterion (rather than the actual stress). Three different damage representations (one without and two with the MCR-effect) are outlined in Section 3. All these representations employ a single scalar damage variable. The condition for discontinuous bifurcations (band-shaped localization), for the considered class of models, is given in Section 4 in terms of the critical band direction and the corresponding state. In particular, we give closed-form solutions at plane stress for the standard model of isotropic damage (without MCR-effect). Based on a specific model for metals (including the von Mises yield criterion with nonlinear hardening and a particular damage law), we evaluate in Section 5 the bifurcation results for semi-brittle as well as ductile damage development. In particular, the effect of MCR on the bifurcation results is quantified numerically. The small strain format is employed throughout the paper.

2. A generic formulation for plasticity coupled to damage

In order to restrict the formal complexity of the theoretical framework (yet allowing for useful modelling of the MCR-effect) while adopting a quite general format of plasticity, we propose that the free energy (per unit volume) Ψ is decomposed into elastic and plastic parts as follows[1]

$$\Psi\left(\boldsymbol{\varepsilon}^{\mathrm{e}}, \underline{\kappa}, \alpha\right) = \Psi^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}}, \alpha) + \Psi^{\mathrm{p}}(\underline{\kappa}) \tag{1}$$

where $\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}$ is the elastic part of the strain. We have introduced the following internal variables: the plastic strain $\boldsymbol{\varepsilon}^{\mathrm{p}}$, the hardening variables $\underline{\kappa}$, and the (scalar) damage variable α.[2] The elastic part of the free energy, Ψ^{e}, is supposed to be convex in the elastic strains $\boldsymbol{\varepsilon}^{\mathrm{e}}$ (for any state, represented by α). Moreover, $\Psi^{\mathrm{p}}(\underline{\kappa})$ is a convex function of $\underline{\kappa}$, but otherwise arbitrary (as yet).

From the Clausius–Duhem-Inequality (CDI), the following constitutive relations are obtained:

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}}, \quad D = \boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \underline{K}^{\mathrm{t}}\dot{\underline{\kappa}} + A\dot{\alpha} \geqslant 0 \tag{2}$$

where D is the rate of dissipation. In (2), we introduced the dissipative 'hardening stresses' $\underline{K}$ and 'damage stress' A, that are energy-conjugated to $\underline{\kappa}$ and α as follows:

$$\underline{K} = -\frac{\partial \Psi^{\mathrm{p}}}{\partial \underline{\kappa}}, \quad A = -\frac{\partial \Psi^{\mathrm{e}}}{\partial \alpha} \tag{3}$$

According to the 'equivalent strain principle', cf Lemaitre (1992), we shall also define the 'effective' stress $\hat{\boldsymbol{\sigma}}$ as the stress that would result if no damage whatsoever had occurred, i.e.

$$\hat{\boldsymbol{\sigma}} = \frac{\partial \hat{\Psi}^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} \quad \text{with} \quad \hat{\Psi}^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}}) \stackrel{\mathrm{def}}{=} \Psi^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}}, 0) \tag{4}$$

It appears that $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{\mathrm{e}}, \alpha)$, whereas $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}^{\mathrm{e}})$. We shall now make the physically significant assumption that there exists a one-to-one relationship between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^{\mathrm{e}}$ for given α, i.e. it is formally possible to invert the relationship $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{\mathrm{e}}, \alpha)$ to obtain $\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{\varepsilon}^{\mathrm{e}}(\boldsymbol{\sigma}, \alpha)$. Hence, we may express

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}^{\mathrm{e}}) \stackrel{\mathrm{def}}{=} \hat{\boldsymbol{\sigma}}\,(\boldsymbol{\sigma}, \alpha) \tag{5}$$

and this relationship will be employed subsequently.

According to the widely accepted 'effective stress principle' of Continuum Damage Mechanics, we now replace $\boldsymbol{\sigma}$ by $\hat{\boldsymbol{\sigma}}$ in the yield function for the undamaged behavior. The convex set $\mathscr{B}$ of plastically admissible states (in the generalized stress space of effective stresses and hardening stresses) is then defined as

$$\mathscr{B} = \left\{\left(\hat{\boldsymbol{\sigma}}, \underline{K}\right) \mid \Phi\left(\hat{\boldsymbol{\sigma}}, \underline{K}\right) \leqslant 0\right\} \tag{6}$$

where Φ is the yield function for the undamaged material.[3] Clearly, such a definition of $\mathscr{B}$ relates to the class of 'generalized standard materials', defined by Halphen and Son (1975). However, this class of materials is not sufficiently general for two reasons: (1) non-associative flow rules need to be

[1] This simplification infers decoupling of elastic and plastic characteristics.

[2] Underscore denotes matrix. Hence, $\underline{\kappa}$ contains the components of tensors of even rank.

[3] For simplicity, we assume henceforth that the yield and potential functions are smooth.

accommodated for pressure-sensitive behavior (e.g. involving dilatancy effects). (2) The damage stress A is not included as an argument of Φ, which calls for generalization in formulating the proper damage law, that couples kinetically the development of damage to that of inelastic deformation, cf Lemaitre (1992), Johansson and Runesson (1997).

To allow for nonassociative plastic flow and hardening rules we introduce the dissipative potential Φ^* such that

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\mu}\frac{\partial \Phi^*}{\partial \boldsymbol{\sigma}} = \dot{\mu}\boldsymbol{\varphi}^*:\mathscr{M} \quad \text{with} \quad \boldsymbol{\varphi}^* \stackrel{\text{def}}{=} \frac{\partial \Phi^*}{\partial \hat{\boldsymbol{\sigma}}} \quad \text{and} \quad \mathscr{M} \stackrel{\text{def}}{=} \frac{\partial \hat{\boldsymbol{\sigma}}}{\partial \boldsymbol{\sigma}} \tag{7}$$

where $\dot{\mu}$ is the plastic multiplier. Clearly, one should choose Φ^* in such a fashion that the CDI is satisfied, i.e. that thermodynamic admissibility is achieved. In order to define the fourth-order transformation $\mathscr{M}$, we first establish the rate form of (2) as

$$\dot{\boldsymbol{\sigma}} = \hat{\mathscr{E}}^{\mathrm{e}}:\boldsymbol{\varepsilon}^{\mathrm{e}} - \boldsymbol{\beta}\dot{\alpha} \quad \text{with} \quad \hat{\mathscr{E}}^{\mathrm{e}} \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = \frac{\partial^2 \Psi^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}} \otimes \partial \boldsymbol{\varepsilon}^{\mathrm{e}}}, \quad \boldsymbol{\beta} \stackrel{\text{def}}{=} -\frac{\partial \boldsymbol{\sigma}}{\partial \alpha} = -\frac{\partial^2 \Psi^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}\, \partial \alpha} \tag{8}$$

From the definition of $\hat{\boldsymbol{\sigma}}$ in (4) follows

$$\dot{\hat{\boldsymbol{\sigma}}} = \mathscr{E}^{\mathrm{e}}:\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} \quad \text{with} \quad \mathscr{E}^{\mathrm{e}} \stackrel{\text{def}}{=} \frac{\partial \hat{\boldsymbol{\sigma}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = \frac{\partial^2 \hat{\Psi}^{\mathrm{e}}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}} \otimes \partial \boldsymbol{\varepsilon}^{\mathrm{e}}} \tag{9}$$

Upon comparing (8) and (9), we conclude that

$$\dot{\hat{\boldsymbol{\sigma}}} = \mathscr{M}:\dot{\boldsymbol{\sigma}} + \mathscr{M}:\boldsymbol{\beta}\dot{\alpha} \quad \text{with} \quad \mathscr{M} = \mathscr{E}^{\mathrm{e}}:\left(\hat{\mathscr{E}}^{\mathrm{e}}\right)^{-1} \tag{10}$$

Remark: Since $\mathscr{E}^{\mathrm{e}}$ and $\hat{\mathscr{E}}^{\mathrm{e}}$ are not necessarily coaxial in the general situation, nothing can be said generally about the (possible) major symmetry of $\mathscr{M}$. •

The hardening rules are introduced in the usual fashion as

$$\dot{\underline{\kappa}} = \dot{\mu}\frac{\partial \Phi^*}{\partial \underline{K}} \tag{11}$$

and we adopt the loading (Kuhn–Tucker complementary) conditions

$$\dot{\mu} \geqslant 0, \quad \Phi\left(\hat{\boldsymbol{\sigma}}, \underline{K}\right) \leqslant 0, \quad \dot{\mu}\Phi\left(\hat{\boldsymbol{\sigma}}, \underline{K}\right) = 0 \tag{12}$$

In order to complete the model concept, we choose the damage rule as

$$\dot{\alpha} = \dot{\mu}\frac{\partial \Upsilon}{\partial A} \tag{13}$$

where $\Upsilon(A, \alpha)$ is a positive scalar function that is monotonically increasing in both its arguments.

By combining (2), (3) with (7), (11), we obtain the constitutive relations

$$\dot{\hat{\boldsymbol{\sigma}}} = \dot{\hat{\boldsymbol{\sigma}}}^{\mathrm{e}} - \dot{\mu}\mathscr{E}^{\mathrm{e}}:\boldsymbol{\varphi}^*:\mathscr{M} \quad \text{with} \quad \dot{\hat{\boldsymbol{\sigma}}}^{\mathrm{e}} = \mathscr{E}^{\mathrm{e}}:\dot{\boldsymbol{\varepsilon}} \tag{14}$$

$$\dot{\underline{K}} = -\dot{\mu}\underline{H}\frac{\partial \Phi^*}{\partial \underline{K}} \quad \text{with} \quad \underline{H} = \frac{\partial^2 \Psi^{\mathrm{p}}}{\partial \underline{\kappa} \partial \underline{\kappa}} \tag{15}$$

We next give the corresponding continuum tangent stiffness (CTS) tensor for the strain-controlled format. Provided $\Phi = 0$, we distinguish between plastic loading (L) and elastic unloading (U) via the (usual) conditions

$$\boldsymbol{\varphi}:\mathscr{E}^{\mathrm{e}}:\dot{\boldsymbol{\varepsilon}} > 0(\mathrm{L}) \quad \boldsymbol{\varphi}:\mathscr{E}^{\mathrm{e}}:\dot{\boldsymbol{\varepsilon}} \leqslant 0\,(\mathrm{U}) \quad \text{with } \boldsymbol{\varphi} \stackrel{\mathrm{def}}{=} \frac{\partial \Phi}{\partial \hat{\boldsymbol{\sigma}}} \tag{16}$$

and we first obtain

$$\dot{\mu} = \frac{1}{h}\boldsymbol{\varphi}:\mathscr{E}^{\mathrm{e}}:\dot{\boldsymbol{\varepsilon}} > 0\,(\mathrm{L}), \quad \dot{\mu} = 0\,(\mathrm{U}) \tag{17}$$

where the modulus $h > 0$ is defined as

$$h = \boldsymbol{\varphi}:\check{\mathscr{E}}^{\mathrm{e}}:\boldsymbol{\varphi}^* + \bar{H} \quad \text{with } \bar{H} = \left(\frac{\partial \Phi}{\partial \underline{K}}\right)^t \underline{H}\frac{\partial \Phi^*}{\partial \underline{K}}, \quad \check{\mathscr{E}}^{\mathrm{e}} = \mathscr{E}^{\mathrm{e}}:\left(\hat{\mathscr{E}}^{\mathrm{e}}\right)^{-1}:\mathscr{E}^{\mathrm{e}} \tag{18}$$

Hence, we obtain $\mathscr{E}^{\mathrm{ep}}$ in the relation

$$\dot{\hat{\boldsymbol{\sigma}}} = \mathscr{E}^{\mathrm{ep}}:\dot{\boldsymbol{\varepsilon}} \quad \text{with } \mathscr{E}^{\mathrm{ep}} = \mathscr{E}^{\mathrm{e}} - \frac{1}{h}\check{\mathscr{E}}^{\mathrm{e}}:\boldsymbol{\varphi}^* \otimes \boldsymbol{\varphi}:\mathscr{E}^{\mathrm{e}} \ (\mathrm{L}) \tag{19}$$

$$\dot{\hat{\boldsymbol{\sigma}}} = \mathscr{E}^{\mathrm{e}}:\dot{\boldsymbol{\varepsilon}}(\mathrm{U}) \tag{20}$$

Upon using the relation (2), we obtain the CTS-tensor $\hat{\mathscr{E}}^{\mathrm{ep}}$ in the relation

$$\dot{\boldsymbol{\sigma}} = \hat{\mathscr{E}}^{\mathrm{ep}}:\dot{\boldsymbol{\varepsilon}} \quad \text{with } \hat{\mathscr{E}}^{\mathrm{ep}} = \hat{\mathscr{E}}^{\mathrm{e}} - \frac{1}{h}\mathscr{E}^{\mathrm{e}}:\hat{\boldsymbol{\varphi}}^* \otimes \boldsymbol{\varphi}:\mathscr{E}^{\mathrm{e}} \ (\mathrm{L}) \tag{21}$$

$$\dot{\boldsymbol{\sigma}} = \hat{\mathscr{E}}^{\mathrm{e}}:\dot{\boldsymbol{\varepsilon}} \ (\mathrm{U}) \tag{22}$$

where we have introduced the 'effective flow direction' $\hat{\boldsymbol{\varphi}}^*$ as

$$\hat{\boldsymbol{\varphi}}^* \stackrel{\mathrm{def}}{=} \boldsymbol{\varphi}^* + (\mathscr{E}^{\mathrm{e}})^{-1}:\boldsymbol{\beta}\frac{\partial \Upsilon}{\partial A} \tag{23}$$

Remark: The 'major' nonsymmetry of $\hat{\mathscr{E}}^{\mathrm{ep}}$ stems from two sources: (a) the nonassociative flow rule and (b) the damage rule. •

As special cases of the general theory, we consider below three different models of damage, which are all based on isotropic elasticity for the virgin material, i.e.,

$$\hat{\Psi}^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}}) = G \mid \boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{dev}} \mid^2 + \frac{K_{\mathrm{b}}}{2}\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)^2 = G \mid \boldsymbol{\varepsilon}^{\mathrm{e}} \mid^2 + L\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)^2 \tag{24}$$

where G, K_{b} (and $L = K_{\mathrm{b}} - 2G/3$) define the constant elastic moduli.

3. Specific damage representations

3.1. Model 1: Classical damage format (without MCR-effect)

The simplest possible damage representation, without any MCR-effect, can be described upon introducing the elastic 'damaged-strain' $\breve{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ as follows:

$$\breve{\boldsymbol{\varepsilon}}^{\mathrm{e}} \stackrel{\mathrm{def}}{=} (1-\alpha)^{1/2}\boldsymbol{\varepsilon}^{\mathrm{e}} \tag{25}$$

whereby Ψ^{e} is chosen as

$$\Psi^{\mathrm{e}} = \hat{\Psi}^{\mathrm{e}}(\breve{\boldsymbol{\varepsilon}}^{\mathrm{e}}) = (1-\alpha)\hat{\Psi}^{\mathrm{e}}(\boldsymbol{\varepsilon}^{\mathrm{e}}) \tag{26}$$

This gives the stress $\boldsymbol{\sigma}$ and the 'damage stress' A as

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = (1-\alpha)\hat{\boldsymbol{\sigma}} \quad \text{with} \quad \hat{\boldsymbol{\sigma}} = 2G\boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{dev}} + K_{\mathrm{b}}\boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{vol}}\boldsymbol{\delta} \tag{27}$$

$$A = -\frac{\partial \Psi}{\partial \alpha} = \frac{1}{6G}\hat{\sigma}_{\mathrm{e}}^2 + \frac{1}{2K_{\mathrm{b}}}\hat{\sigma}_{\mathrm{m}}^2 \quad \text{with } \hat{\sigma}_{\mathrm{e}} = \sqrt{\frac{3}{2}}\,|\,\hat{\boldsymbol{\sigma}}_{\mathrm{dev}}\,|\,, \quad \hat{\sigma}_{\mathrm{m}} = \frac{1}{3}\hat{\boldsymbol{\sigma}}:\boldsymbol{\delta} \tag{28}$$

The CTS-tensor $\hat{\mathscr{E}}^{\mathrm{e}}$ is given as

$$\hat{\mathscr{E}}^{\mathrm{e}} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = (1-\alpha)\mathscr{E}^{\mathrm{e}} \quad \text{with } \mathscr{E}^{\mathrm{e}} = 2G\mathscr{I}_{\mathrm{dev}} + K_{\mathrm{b}}\boldsymbol{\delta}\otimes\boldsymbol{\delta} \tag{29}$$

Here, $\mathscr{I}_{\mathrm{dev}} = \mathscr{I} - \frac{1}{3}\boldsymbol{\delta}\otimes\boldsymbol{\delta}$ is the deviatoric identity tensor that projects any second-order tensor onto its deviator, where $\mathscr{I} = \frac{1}{2}(\boldsymbol{\delta}\,\overline{\otimes}\,\boldsymbol{\delta} + \boldsymbol{\delta}\,\underline{\otimes}\,\boldsymbol{\delta})$ is the (symmetric) fourth-order identity tensor.[4]

Finally $\boldsymbol{\beta}$ is defined as

$$\boldsymbol{\beta} = -\frac{\partial \boldsymbol{\sigma}}{\partial \alpha} = \hat{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1-\alpha} \tag{30}$$

Let us next consider the relations that are pertinent to plane stress, which is defined by $\dot{\sigma}_{i3}=0$, $i=1,2,3$, whereby the coordinates x_1, x_2 are located in the plane of interest. Since $\sigma_{i3} = \hat{\sigma}_{i3} = \beta_{i3} = 0$ from (30), it follows that we may part-invert $\hat{\mathscr{E}}^{\mathrm{e}}$ in (29) to obtain the plane–stress version[5]

$$\hat{\mathscr{E}}^{\mathrm{e}} = 2G(1-\alpha)\left(\mathscr{I} + \frac{\nu}{1-\nu}\boldsymbol{\delta}\otimes\boldsymbol{\delta}\right), \quad \left(\hat{\mathscr{E}}^{\mathrm{e}}\right)^{-1} \stackrel{\mathrm{def}}{=} \frac{1}{2G(1-\alpha)}\left(\mathscr{I} - \frac{\nu}{1+\nu}\boldsymbol{\delta}\otimes\boldsymbol{\delta}\right) \tag{31}$$

in complete analogy with the situation without damage.

We also obtain

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[4] For second-order tensors $\mathbf{u}$ and $\mathbf{v}$, we introduce the open products $(\mathbf{u}\,\overline{\otimes}\,\mathbf{v})_{abcd} = u_{ac}v_{bd}$, $(\mathbf{u}\,\underline{\otimes}\,\mathbf{v})_{abcd} = u_{ad}v_{bc}$.

[5] With Greek indices referring to in-plane components, the Cartesian components of $\hat{\mathscr{E}}^{\mathrm{e}}$ are

$$\hat{\mathscr{E}}^{\mathrm{e}}_{\alpha\beta\gamma\delta} = 2\hat{G}\left(\mathscr{I}_{\alpha\beta\gamma\delta} + \frac{\nu}{1-\nu}\delta_{\alpha\beta}\delta_{\gamma\delta}\right) \quad \text{with } \mathscr{I}_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$$

Moreover, subscript 'v' indicates 'volumetric' in-plane, e.g. $\varepsilon_v = \varepsilon_{\alpha\alpha}$. The true volumetric component is indicated by subscript 'vol', e.g. $\varepsilon_{\mathrm{vol}} = \varepsilon_{kk}$.

$$\hat{\boldsymbol{\varphi}}^* = \boldsymbol{\varphi}^* + \boldsymbol{\varphi}^{\mathrm{d}} \quad \text{with } \boldsymbol{\varphi}^{\mathrm{d}} \stackrel{\mathrm{def}}{=} \frac{1}{2G}\left(\hat{\boldsymbol{\sigma}} - \frac{3\nu}{1+\nu}\hat{\sigma}_{\mathrm{m}}\boldsymbol{\delta}\right)\frac{\partial \Upsilon}{\partial A} \tag{32}$$

3.2. Model 2: MCR-effect in isotropic part of the elastic strain

A simplified form of MCR is restricted to isotropic compression and extension, cf Yazdani and Schreyer (1988), whereas deviatoric loading is assumed to always produce damage. Such a simple model is justified when microcracking is not associated with any directionality, which may be of relevance for a polycrystalline microstructure with randomly oriented crack planes. Let us then introduce the 'damaged' elastic portion of the strain tensor $\check{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ as follows:

$$\check{\boldsymbol{\varepsilon}}^{\mathrm{e}} = (1-\alpha)^{1/2}\boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{dev}} + \tfrac{1}{3}\left(1 - g\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)\alpha\right)^{1/2}\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\boldsymbol{\delta} \tag{33}$$

where $g(x)$ is taken as a (smooth) monotonically increasing function such that

$$g(\infty) = 1, \quad g(-\infty) = g_0, \quad 0 \leqslant g_0 \leqslant 1 \tag{34}$$

Example 1: A typical example would be

$$g(x) = \frac{1+g_0}{2} + \frac{1-g_0}{\pi}\arctan\left(\frac{x-x_0}{x_{\mathrm{R}}}\right), \quad x_{\mathrm{R}} > 0 \tag{35}$$

which is depicted in Fig. 1(a) The 'steepness' of this function is controlled by the (reference) value x_{R}, whereas the offset value x_0 represents the sensitivity for compression.

Example 2: A simplified version of (35), which has been suggested by Lemaitre and Chaboche (1990), Klarbring and Lundin (1997), is defined as

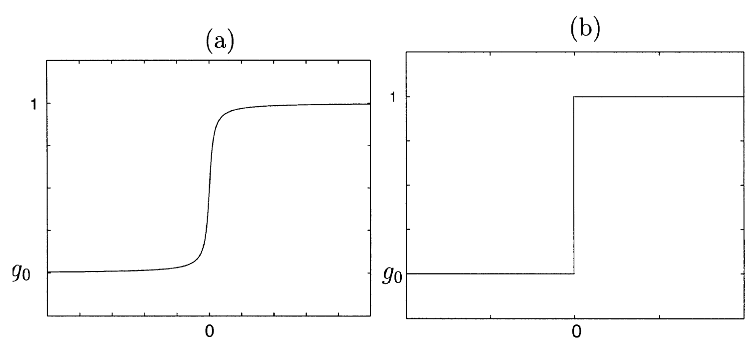

Fig. 1. (a) Smooth and (b) discontinuous MCR-functions.

$$g(x) = g_0 + (1 - g_0)\mathscr{H}(x) \tag{36}$$

where $\mathscr{H}(x)$ is the Heaviside function,[6] which is shown in Fig. 1(b).

The elastic part of the free energy is now chosen as

$$\Psi^{\mathrm{e}} = G \mid \check{\boldsymbol{\varepsilon}}^{\mathrm{e}}_{\mathrm{dev}} \mid^2 + \frac{K_{\mathrm{b}}}{2}\left(\check{\varepsilon}^{\mathrm{e}}_{\mathrm{vol}}\right)^2 = G(1 - \alpha) \mid \boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{dev}} \mid^2 + \frac{K_{\mathrm{b}}}{2}\left(1 - g\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)\alpha\right)\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)^2 \tag{37}$$

from which the nominal stress $\boldsymbol{\sigma}$ and the 'damage' stress A are obtained:

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = 2G(1 - \alpha)\boldsymbol{\varepsilon}^{\mathrm{e}}_{\mathrm{dev}} + K_{\mathrm{b}}\left(1 - g_1\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)\alpha\right)\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\boldsymbol{\delta} \tag{38}$$

$$A = -\frac{\partial \Psi}{\partial \alpha} = \frac{1}{6G}\hat{\sigma}^2_{\mathrm{e}} + \frac{1}{2K_{\mathrm{b}}} g\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)\hat{\sigma}^2_{\mathrm{m}} \tag{39}$$

where the function $g_1(x)$ is defined as

$$g_1(x) = g(x) + \tfrac{1}{2}xg'(x) \tag{40}$$

Remark: In (39), we may express g in terms of $\hat{\sigma}_{\mathrm{m}}$ via the simple change of arguments $\varepsilon^{\mathrm{e}}_{\mathrm{vol}} = \hat{\sigma}_{\mathrm{m}}/K_{\mathrm{b}}$. •

As to $\hat{\mathscr{E}}^{\mathrm{e}}$, we obtain

$$\hat{\mathscr{E}}^{\mathrm{e}} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = 2G(1 - \alpha)\mathscr{I}_{\mathrm{dev}} + K_{\mathrm{b}}\left(1 - g_2\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)\alpha\right)\boldsymbol{\delta} \otimes \boldsymbol{\delta} \tag{41}$$

where the function $g_2(x)$is defined as

$$g_2(x) = g_1(x) + xg_1'(x) \tag{42}$$

Finally, we obtain

$$\boldsymbol{\beta} = -\frac{\partial \boldsymbol{\sigma}}{\partial \alpha} = \hat{\boldsymbol{\sigma}}_{\mathrm{dev}} + g_1\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)\hat{\sigma}_{\mathrm{m}}\boldsymbol{\delta} = \frac{\boldsymbol{\sigma}_{\mathrm{dev}}}{1 - \alpha} + \frac{g_1\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)}{1 - g_1\left(\varepsilon^{\mathrm{e}}_{\mathrm{vol}}\right)\alpha}\sigma_{\mathrm{m}}\boldsymbol{\delta} \tag{43}$$

Special case: When the special case of $g(x)$ in (36) is chosen, then $g(x) = g_1(x) = g_2(x)$, since $g'(x) = 0, \forall x \neq 0$.

We next turn to plane stress. Even in this case it is possible to obtain a simple analytic expression for $\hat{\mathscr{E}}^{\mathrm{e}}$, corresponding to that in (31). However, to avoid unnecessary technical complexity, we omit these expressions here and simply conclude that the pertinent expression can be obtained by numerical part-inversion of the relation in (41). Likewise, the appropriate plane stress version of $\hat{\mathscr{E}}^{\mathrm{ep}}$ in (21) is conveniently obtained in a numerical fashion.

Remark: For the present model $\hat{\sigma}_{33} \neq 0$ (although $\sigma_{33} = 0$) at plane stress, unlike the case for the classical Model 1. •

[6] $\mathscr{H}(x) = 1$ if $x \geqslant 0$, $\mathscr{H}(x) = 0$ if $x < 0$.

3.3. Model 3: MCR-effect in principal elastic strains

A model that represents a simplified version of the model by Ramtani (1990), is defined by the following choice of the 'damaged' elastic strain $\check{\boldsymbol{\varepsilon}}^{\mathrm{e}}$:

$$\check{\boldsymbol{\varepsilon}}^{\mathrm{e}} \stackrel{\mathrm{def}}{=} \sum_{i=1}^{3} \check{\varepsilon}_i^{\mathrm{e}} \boldsymbol{g}_i \quad \text{with } \check{\varepsilon}_i^{\mathrm{e}} = \left(1 - g\left(\varepsilon_i^{\mathrm{e}}\right)\alpha\right)^{1/2} \varepsilon_i^{\mathrm{e}} \tag{44}$$

where $\check{\varepsilon}_i^{\mathrm{e}}$ are the principal values of $\check{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ w.r.t. the eigendyad $\boldsymbol{g}_i$ of $\boldsymbol{\varepsilon}^{\mathrm{e}}$, and $g(x)$ is (still) the regularization function with the properties in (34). Clearly, this choice relates the MCR-effect to the sign of the principal elastic strains. For example, this model admits microcrack opening in the transverse direction at uniaxial compression (splitting mode). If $\varepsilon_i^{\mathrm{e}}$ are distinct,[7] then we may use Serrin's formula to represent $\boldsymbol{g}_i$ as follows:

$$\boldsymbol{g}_i = \frac{1}{d_i}\left((\boldsymbol{\varepsilon}^{\mathrm{e}})^2 - \left(I_1 - \varepsilon_i^{\mathrm{e}}\right)\boldsymbol{\varepsilon}^{\mathrm{e}} + \frac{J}{\varepsilon_i^{\mathrm{e}}}\boldsymbol{\delta} \right) \tag{45}$$

where d_i is defined as

$$d_i \stackrel{\mathrm{def}}{=} 2\left(\varepsilon_i^{\mathrm{e}}\right)^2 - I_1 \varepsilon_i^{\mathrm{e}} + J\left(\varepsilon_i^{\mathrm{e}}\right)^{-1}, \quad I_1 = \sum_{i=1}^{3} \varepsilon_i^{\mathrm{e}}, \quad J = \varepsilon_1^{\mathrm{e}} \varepsilon_2^{\mathrm{e}} \varepsilon_3^{\mathrm{e}} \tag{46}$$

The elastic part of the free energy now becomes:

$$\Psi^{\mathrm{e}} = G|\check{\boldsymbol{\varepsilon}}^{\mathrm{e}}|^2 + \frac{L}{2}\left(\check{\varepsilon}_{\mathrm{vol}}^{\mathrm{e}}\right)^2 = G \sum_{i=1}^{3}\left(\check{\varepsilon}_i^{\mathrm{e}}\right)^2 + \frac{L}{2}\left(\sum_{i=1}^{3} \check{\varepsilon}_i^{\mathrm{e}}\right)^2 \tag{47}$$

from which the nominal stress $\boldsymbol{\sigma}$ and the 'damage' stress A are given as:

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = \sum_{i=1}^{3} \sigma_i\left(\varepsilon_i^{\mathrm{e}}, \alpha\right)\boldsymbol{g}_i \quad \text{with } \sigma_i = \frac{\partial \Psi}{\partial \check{\varepsilon}_i^{\mathrm{e}}} \frac{\partial \check{\varepsilon}_i^{\mathrm{e}}}{\partial \varepsilon_i^{\mathrm{e}}} \tag{48}$$

$$A = -\frac{\partial \Psi}{\partial \alpha} = \sum_{i=1}^{3} A_i \quad \text{with } A_i = -\frac{\partial \Psi}{\partial \check{\varepsilon}_i^{\mathrm{e}}} \frac{\partial \check{\varepsilon}_i^{\mathrm{e}}}{\partial \alpha} \tag{49}$$

where

$$\frac{\partial \Psi}{\partial \check{\varepsilon}_i^{\mathrm{e}}} = 2G\check{\varepsilon}_i^{\mathrm{e}} + L\sum_{j=1}^{3} \check{\varepsilon}_j^{\mathrm{e}}, \quad \frac{\partial \check{\varepsilon}_i^{\mathrm{e}}}{\partial \varepsilon_i^{\mathrm{e}}} = \frac{1 - g_1\left(\varepsilon_i^{\mathrm{e}}\right)\alpha}{\left(1 - g\left(\varepsilon_i^{\mathrm{e}}\right)\alpha\right)^{1/2}}, \quad \frac{\partial \check{\varepsilon}_i^{\mathrm{e}}}{\partial \alpha} = -\frac{\varepsilon_i^{\mathrm{e}} g\left(\varepsilon_i^{\mathrm{e}}\right)}{2\left(1 - g\left(\varepsilon_i^{\mathrm{e}}\right)\alpha\right)^{1/2}} \tag{50}$$

It is emphasized that $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^{\mathrm{e}}$ are coaxial and that $\boldsymbol{\sigma}$ is an isotropic function of $\boldsymbol{\varepsilon}^{\mathrm{e}}$.

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[7] A slight perturbation is made whenever $\varepsilon_i^{\mathrm{e}}$ are not distinct.

As to the computation of $\hat{\mathscr{E}}^{\mathrm{e}}$, we obtain

$$\hat{\mathscr{E}}^{\mathrm{e}} = \sum_{i=1}^{3} \gamma_i (\boldsymbol{g}_i \otimes \boldsymbol{g}_i) + \sigma_i \mathscr{G}_i \quad \text{with } \mathscr{G}_i \stackrel{\text{def}}{=} \frac{\partial g_i}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} \tag{51}$$

where the coefficients $\gamma_i(\varepsilon_i^{\mathrm{e}}, \alpha)$ are given as

$$\gamma_i = (2G + L)\left(\frac{\partial \check{\varepsilon}_i^{\mathrm{e}}}{\partial \varepsilon_i^{\mathrm{e}}}\right)^2 + \frac{\partial \Psi}{\partial \check{\varepsilon}_i^{\mathrm{e}}} g_3(\alpha, \varepsilon_i^{\mathrm{e}}) \tag{52}$$

with

$$g_3(\alpha, \varepsilon_i^{\mathrm{e}}) = \frac{\frac{1}{2}\alpha g'(\varepsilon_i^{\mathrm{e}})(1 - g_1(\varepsilon_i^{\mathrm{e}})\alpha) - \alpha g_1'(\varepsilon_i^{\mathrm{e}})(1 - g(\varepsilon_i^{\mathrm{e}})\alpha)}{(1 - g(\varepsilon_i^{\mathrm{e}})\alpha)^{3/2}} \tag{53}$$

After some manipulations, we obtain:

$$\mathscr{G}_i = \frac{1}{d_i}\left(2\mathscr{I}_{\varepsilon^{\mathrm{e}}} - (I_1 - \varepsilon_i^{\mathrm{e}})\mathscr{I} + \sum_{j=1}^{3}(I_1 - \varepsilon_i^{\mathrm{e}} - 2\varepsilon_j^{\mathrm{e}})\boldsymbol{g}_j \otimes \boldsymbol{g}_j\right) \tag{54}$$

where we introduced the fourth-order tensor

$$\mathscr{I}_{\boldsymbol{\varepsilon}^{\mathrm{e}}} \stackrel{\text{def}}{=} \frac{1}{4}(\boldsymbol{\delta} \mathbin{\bar{\otimes}} \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{e}} \mathbin{\bar{\otimes}} \boldsymbol{\delta} + \boldsymbol{\delta} \mathbin{\underline{\otimes}} \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{e}} \mathbin{\underline{\otimes}} \boldsymbol{\delta}) \tag{55}$$

Remark: It appears that $\mathscr{I}_{\boldsymbol{\varepsilon}^{\mathrm{e}}}$ is the generalization of $\mathscr{I} = \mathscr{I}_{\boldsymbol{\delta}}$. •

Finally $\boldsymbol{\beta}$ is defined as

$$\boldsymbol{\beta} = -\frac{\partial \boldsymbol{\sigma}}{\partial \alpha} = -\sum_{j=1}^{3} \frac{\partial \Psi}{\partial \check{\varepsilon}_i^{\mathrm{e}}} \frac{\partial^2 \check{\varepsilon}_i^{\mathrm{e}}}{\partial \varepsilon_i^{\mathrm{e}} \partial \alpha} + \frac{\partial^2 \Psi}{\partial \check{\varepsilon}_i^{\mathrm{e}} \partial \alpha} \frac{\partial \check{\varepsilon}_i^{\mathrm{e}}}{\partial \varepsilon_i^{\mathrm{e}}} \tag{56}$$

where

$$\frac{\partial^2 \Psi}{\partial \varepsilon_i^{\mathrm{e}} \partial \alpha} = 2G \frac{\partial \check{\varepsilon}_i^{\mathrm{e}}}{\partial \alpha} + L \sum_{j=1}^{3} \frac{\partial \check{\varepsilon}_j^{\mathrm{e}}}{\partial \alpha}, \quad \frac{\partial^2 \check{\varepsilon}_i^{\mathrm{e}}}{\partial \varepsilon_i^{\mathrm{e}} \partial \alpha} = \frac{1}{2} \frac{g(\varepsilon_i^{\mathrm{e}}) - 2g_1(\varepsilon_i^{\mathrm{e}}) + g(\varepsilon_i^{\mathrm{e}}) g_1(\varepsilon_i^{\mathrm{e}})\alpha}{(1 - g(\varepsilon_i^{\mathrm{e}})\alpha)^{3/2}} \tag{57}$$

In the case of plane stress, we use numerical part-inversion to obtain the pertinent plane stress versions of $\hat{\mathscr{E}}^{\mathrm{e}}$ and $\hat{\mathscr{E}}^{\mathrm{ep}}$.

4. Bifurcation of incremental solution

4.1. Preliminaries

In order to assess the possibility for bifurcation of the incremental solution at a certain (thermodynamic) state, we may put forward arguments in close agreement with those of Runesson et al. (1991). Hence, we shall examine the spectral properties of the characteristic tensor $\hat{\boldsymbol{Q}}^{\mathrm{ep}}$ defined as

$$\hat{\boldsymbol{Q}}^{\mathrm{ep}} = \boldsymbol{n} \cdot \hat{\mathscr{E}}^{\mathrm{ep}} \cdot \boldsymbol{n} = \hat{\mathbf{Q}}^{\mathrm{e}} - \frac{1}{h}\hat{\boldsymbol{a}}^{*} \otimes \boldsymbol{a} \quad \text{with } \hat{\boldsymbol{Q}}^{\mathrm{e}} = \boldsymbol{n} \cdot \hat{\mathscr{E}}^{\mathrm{e}} \cdot \boldsymbol{n} \tag{58}$$

where $\boldsymbol{n}$ is the unit normal vector of the characteristic surface. The vectors $\boldsymbol{a}(\boldsymbol{n})$ and $\hat{\boldsymbol{a}}^{*}(\boldsymbol{n})$ are given as

$$\boldsymbol{a} = \boldsymbol{\varphi}:\mathscr{E}^{\mathrm{e}} \cdot \boldsymbol{n}, \quad \hat{\boldsymbol{a}}^{*} = \hat{\boldsymbol{\varphi}}^{*}:\mathscr{E}^{\mathrm{e}} \cdot \boldsymbol{n} \tag{59}$$

The smallest eigenvalue $\lambda^{(1)}$ of $\hat{\boldsymbol{Q}}^{\mathrm{ep}}$, with respect to the metric defined by the 'damaged' elastic tensor $\hat{\boldsymbol{Q}}^{\mathrm{e}} = \boldsymbol{n} \cdot \hat{\mathscr{E}}^{\mathrm{e}} \cdot \boldsymbol{n}$, is given as

$$\lambda^{(1)}(\boldsymbol{n}) = 1 - \frac{1}{h} Y(\boldsymbol{n}) \quad \text{with } h = Z + \bar{H} \tag{60}$$

where we have introduced the abbreviated notation

$$Y(\boldsymbol{n}) = \boldsymbol{a}(\boldsymbol{n}) \cdot \hat{\boldsymbol{P}}^{\mathrm{e}}(\boldsymbol{n}) \cdot \hat{\boldsymbol{a}}^{*}(\boldsymbol{n}), \quad Z = \boldsymbol{\varphi}:\check{\mathscr{E}}^{\mathrm{e}}:\boldsymbol{\varphi}^{*} \tag{61}$$

and $\hat{\boldsymbol{P}}^{\mathrm{e}} = (\hat{\mathbf{Q}}^{\mathrm{e}})^{-1}$. At any given thermodynamic state, s, we define the localization direction $\bar{\boldsymbol{n}}(s)$ such that

$$\bar{\boldsymbol{n}} = \arg\left[\min_{|\boldsymbol{n}|=1} \lambda^{(1)}(\boldsymbol{n})\right] = \arg\left[\max_{|\boldsymbol{n}|=1} Y(\boldsymbol{n})\right] \tag{62}$$

where the relation (60) was used. The critical state, s_{cr}, corresponding to the critical localization direction $\boldsymbol{n}_{\mathrm{cr}} = \bar{\boldsymbol{n}}(s_{\mathrm{cr}})$, must satisfy the relation $\lambda^{(1)}(\bar{\boldsymbol{n}}) = 0$, i.e.

$$\bar{H} + Z - Y(\bar{\boldsymbol{n}}) = 0 \tag{63}$$

where it is realized that $\bar{H}$, Y and Z are, in general, functions of the state (apart from $\bar{\boldsymbol{n}}$)[8]. Eqn (63) is not sufficient in general to determine the critical state s_{cr}. Rather, this state is achieved along a (given) loading path in stress, strain or mixed stress–strain space. After integration of the pertinent constitutive relations, we may check whether the relation (63) is satisfied. If that is the case, the critical state s_{cr} is found, and $\boldsymbol{n}_{\mathrm{cr}}$ can be computed. It is emphasized, once again, that the critical direction $\boldsymbol{n}_{\mathrm{cr}}$ depends, generally, on the loading path.

Depending on the actual constitutive properties, three principally different situations may be encountered:

Situation 1: The first situation is that bifurcation is possible at the very onset of yielding (and development of damage), i.e. when $\alpha = 0$. The corresponding value of $\bar{H}_{\mathrm{cr}}$ is denoted $\bar{H}_{\mathrm{cr},0}$ and it is obtained directly from (63) with $\alpha = 0$ and with known stress state at initial yielding. We remark that no integration of the constitutive relations will be necessary in this particular situation, which is typical for semi-brittle response.

Situation 2. The second situation is defined by $\bar{H}_0 < \bar{H}_{\mathrm{cr},0}$, where we used the notation $\bar{H}_0$ for the initial value of $\bar{H}$ (at the onset of yielding). This means that the critical state is traversed already at the onset of yielding, and it is possible to find many solutions $\boldsymbol{n}$ to (63) such that $\lambda^{(1)}(\boldsymbol{n}) = 0$ [without satisfying (62)].

[8] We should write $\bar{H} = \bar{H}(s)$, $Y = Y(s, \bar{\boldsymbol{n}})$ and $Z = Z(s)$.

Situation 3: The third situation is characterized by $\bar{H}_0 > \bar{H}_{\mathrm{cr},0}$ and $\bar{H}$ continuously decreasing along the considered loading path. As a special case, $\bar{H}$ is constant (like for linear isotropic hardening of the von Mises yield surface). It is then possible to obtain real solutions $\bar{H}_{\mathrm{cr}}$ and α_{cr} only after development of damage ($\alpha_{\mathrm{cr}} > 0$), which is typical for ductile response.

4.2. Model 1: Solution for plane stress

It is possible to establish closed-form expressions for $\boldsymbol{n}_{\mathrm{cr}}$, H_{cr} and α_{cr} from (62) and (63) for this model as direct extensions of those of plasticity without damage, given by Runesson et al. (1991). The pertinent developments are reviewed briefly here.

From (31) it follows that

$$\hat{\boldsymbol{Q}}^{\mathrm{e}} = (1-\alpha)G\left(\boldsymbol{\delta} + \frac{1+\nu}{1-\nu}\boldsymbol{n}\otimes\boldsymbol{n}\right), \quad \hat{\boldsymbol{P}}^{\mathrm{e}} \stackrel{\mathrm{def}}{=} \left(\hat{\boldsymbol{Q}}^{\mathrm{e}}\right)^{-1} = \frac{1}{(1-\alpha)G}\left(\boldsymbol{\delta} - \frac{1+\nu}{2}\boldsymbol{n}\otimes\boldsymbol{n}\right) \tag{64}$$

and

$$\boldsymbol{a} = 2G\left(\boldsymbol{\varphi}\cdot\boldsymbol{n} + \frac{\nu}{1-\nu}\varphi_{\mathrm{v}}\boldsymbol{n}\right), \quad \hat{\boldsymbol{a}}^* = 2G\left(\hat{\boldsymbol{\varphi}}^*\cdot\boldsymbol{n} + \frac{\nu}{1-\nu}\hat{\varphi}_{\mathrm{v}}^*\boldsymbol{n}\right) \tag{65}$$

where φ_{v} and $\hat{\varphi}_{\mathrm{v}}^*$ are the 'volumetric' parts in plane stress of $\boldsymbol{\varphi}$ and $\hat{\boldsymbol{\varphi}}^*$, respectively. These expressions were given in (32).

We shall now make the important assumption that φ and φ^* possess the same principal directions, and that these directions are also identical to those of $\boldsymbol{\sigma}$. Because of the plane stress condition, it follows that two (out of the three) principal directions are located in the plane of interest. The corresponding principal values are labelled φ_1, φ_2 and φ_1^*, φ_2^* (and σ_1, σ_2) in such a way that $\varphi_1 \geqslant \varphi_2$ (without any restriction). However, we assume that this labeling of coordinate axes will also infer that $\varphi_1^* \geqslant \varphi_2^*$ and $\sigma_1 \geqslant \sigma_2$.[9] By (32), we also conclude that $\varphi_1^{\mathrm{d}} \geqslant \varphi_2^{\mathrm{d}}$ and, hence, $\hat{\varphi}_1^* \geqslant \hat{\varphi}_2^*$.

Remark: The magnitude of the out-of-plane components φ_3 and φ_3^* (and $\sigma_3 = 0$) are not related to the magnitude of the in-plane components. •

From $(32)_2$ we now obtain the principal components

$$\varphi_1^{\mathrm{d}} = \frac{1}{E}(\hat{\sigma}_1 - \nu\hat{\sigma}_2)\frac{\partial\Upsilon}{\partial A}, \quad \varphi_2^{\mathrm{d}} = \frac{1}{E}(-\nu\hat{\sigma}_1 + \hat{\sigma}_2)\frac{\partial\Upsilon}{\partial A} \tag{66}$$

where $E = 2G(1+\nu)$.

Subsequently, we shall restrict the analysis to the (main) situation where $\varphi_1 > \varphi_2$ (and $\varphi_1^* > \varphi_2^*$). Following the developments in Runesson et al. (1991), we first define the scalars

$$\hat{c}_\alpha = (1+\nu)\Big[(\varphi_{\mathrm{v}} - \varphi_\alpha)\big(\hat{\varphi}_1^* - \hat{\varphi}_2^*\big) + \big(\hat{\varphi}_{\mathrm{v}}^* - \hat{\varphi}_\alpha^*\big)(\varphi_1 - \varphi_2)\Big], \quad \alpha = 1, 2 \tag{67}$$

[9] This result follows trivially if Φ and Φ^* are isotropic functions of $\hat{\sigma}$.

Remark: It appears that $\hat{c}_\alpha$ are straightforward generalizations of c_α defined for plasticity without damage in Runesson et al. (1991). However, an important difference is that the components are interchanged such that $\hat{c}_2$ and $\hat{c}_1$ generalize c_1 and c_2, respectively, where

$$c_\alpha = \varphi_\alpha(\varphi_1^* - \varphi_2^*) + \varphi_\alpha^*(\varphi_1 - \varphi_2), \quad \alpha = 1, 2 \tag{68}$$

This change of notation is made for convenience in order to facilitate a direct comparison with the more general situation of quasi-isotropic damage development considered in the next section of this paper. •

Three different cases are distinguished:

Case 1: When the conditions $\hat{c}_1 \leqslant 0$ and $\hat{c}_2 \geqslant 0$ are satisfied, then we obtain the critical directions n_1^2 and n_2^2 from

$$n_1^2 = \frac{\hat{c}_2}{\hat{c}_2 - \hat{c}_1}, \quad n_2^2 = -\frac{\hat{c}_1}{\hat{c}_2 - \hat{c}_1} \rightsquigarrow \tan^2\theta_{\mathrm{cr}} = -\frac{\hat{c}_2}{\hat{c}_1} \tag{69}$$

where θ denotes the angle in the x_1x_2-plane from the x_2-axis to the normal vector (n_1, n_2).

Case 2: When $\hat{c}_2 \leqslant 0$ (while $\hat{c}_1 \leqslant 0$), we obtain the solution $\theta_{\mathrm{cr}} = 0°$.

Case 3: When $\hat{c}_1 \geqslant 0$ (while $\hat{c}_2 \geqslant 0$), we obtain $\theta_{\mathrm{cr}} = 90°$.

In each case, we obtain the critical values $\bar{H}_{\mathrm{cr}}$ and α_{cr} from the relation

$$\bar{H} - Y(\alpha_{\mathrm{cr}}) + Z(\alpha_{\mathrm{cr}}) = 0 \tag{70}$$

where

$$Y = \frac{2G}{1-\alpha}\Big[s_1 n_1^2 + s_2 n_2^2 - \gamma^{(2)}(\varphi_1 n_1^2 + \varphi_2 n_2^2)(\hat{\varphi}_1^* n_1^2 + \hat{\varphi}_2^* n_2^2) + \gamma^{(3)}\varphi_{\mathrm{v}}\hat{\varphi}_{\mathrm{v}}^*\Big] \tag{71}$$

$$Z = \frac{2G}{1-\alpha}\Big(\varphi_1\varphi_1^* + \varphi_2\varphi_2^* + \gamma^{(4)}\varphi_{\mathrm{v}}\varphi_{\mathrm{v}}^*\Big) \tag{72}$$

with the notation

$$s_\alpha = 2\varphi_\alpha\hat{\varphi}_\alpha^* + \gamma^{(1)}(\varphi_{\mathrm{v}}\hat{\varphi}_\alpha^* + \varphi_\alpha\hat{\varphi}_{\mathrm{v}}^*), \quad \text{no sum on } \alpha \tag{73}$$

$$\gamma^{(1)} = \nu, \quad \gamma^{(2)} = 1 + \nu, \quad \gamma^{(3)} = \frac{\nu^2}{1-\nu}, \quad \gamma^{(4)} = \frac{\nu}{1-\nu} \tag{74}$$

4.3. Model 2 and Model 3: Solution for plane stress

It is necessary to resort to numerical evaluation of the bifurcation criterion in the general situation (when it is not possible to take advantage of the specific structure of the acoustic tensor). The following algorithm is then employed:

—Compute the pertinent plane stress version of $\hat{\mathscr{E}}^{\mathrm{e}}$ and $\hat{\mathscr{E}}^{\mathrm{ep}}$ by part-inversion. Then, obtain $\hat{\boldsymbol{Q}}^{\mathrm{e}}$ and $\hat{\boldsymbol{Q}}^{\mathrm{ep}}$.

—Compute $\lambda^{(1)}(\boldsymbol{n})$ in (60) and obtain $\bar{\boldsymbol{n}}$ from (62). If $\lambda^{(1)}(\bar{\boldsymbol{n}}) = 0$, then the critical state has been identified and $\boldsymbol{n}_{\mathrm{cr}} = \bar{\boldsymbol{n}}$.

Remark: In fact, closed-form expressions for $\boldsymbol{n}_{\mathrm{cr}}$ can be obtained for Model 2 (similarly to those of Model 1). However, in order to avoid the technical complexity, we refrain from giving the details here. •

5. Specific model for metals

5.1. von Mises plasticity with nonlinear hardening coupled to damage

A model that has been proposed for the analysis of LCF of metals and alloys is based on the von Mises yield surface with nonlinear mixed isotropic and kinematic hardening, originally proposed by Armstrong and Frederick (1966). We then choose

$$\Psi^{\mathrm{p}}(\kappa, \boldsymbol{\beta}) = \tfrac{1}{2} r H \kappa^2 + \tfrac{1}{2}(1-r) H \beta_{\mathrm{e}}^2 \quad \text{with } \beta_{\mathrm{e}} = \sqrt{\tfrac{2}{3}} |\boldsymbol{\beta}_{\mathrm{dev}}| \tag{75}$$

where κ and $\boldsymbol{\beta}$ are the isotropic and kinematic hardening variables, respectively. Moreover H is the initial hardening modulus in uniaxial stress, whereas r is a parameter that controls the relation between isotropic and kinematic hardening: $r = 0$ represents purely kinematic hardening, whereas $r = 1$ represents purely isotropic hardening. The von Mises yield function is defined as

$$\Phi = \hat{\tau}_{\mathrm{e}} - \sigma_{\mathrm{y}} - K, \quad \hat{\tau}_{\mathrm{e}} = \sqrt{\tfrac{3}{2}} |\hat{\boldsymbol{\tau}}_{\mathrm{dev}}| \quad \text{with } \hat{\boldsymbol{\tau}} = \hat{\boldsymbol{\sigma}} - \boldsymbol{B} \tag{76}$$

where we have introduced the dissipative stresses, associated with isotropic and kinematic hardening, as:

$$K = -rH\kappa, \quad \boldsymbol{B} = -\tfrac{2}{3}(1-r)H\boldsymbol{\beta} \tag{77}$$

The potential function Φ^* is chosen as

$$\Phi^* = \Phi + \frac{K^2}{2K_\infty} + \frac{B_{\mathrm{e}}^2}{2B_\infty} \quad \text{with } B_{\mathrm{e}} = \sqrt{\tfrac{3}{2}} |\mathbf{B}_{\mathrm{dev}}| \tag{78}$$

and it appears that $D \geqslant 0$ is satisfied with this choice. The moduli K_∞ and B_∞ are saturation values of isotropic and kinematic hardening, respectively. With (18), we then obtain

$$\bar{H} = rH\left(1 - \frac{K}{K_\infty}\right) + (1-r)H\left(1 - \frac{3}{2\hat{\tau}_{\mathrm{e}} B_\infty} \hat{\boldsymbol{\tau}}_{\mathrm{dev}} : \mathbf{B}_{\mathrm{dev}}\right) \tag{79}$$

To simplify the analysis (but without restricting the generality of the approach), we consider henceforth only isotropic nonlinear hardening, i.e. $r = 1$. Consequently, only monotonic loading paths are investigated. We then obtain from (76) and (78)

$$\varphi_1 = \varphi_1^* = \frac{1}{2\hat{\sigma}_{\mathrm{e}}}(2\hat{\sigma}_1 - \hat{\sigma}_2), \quad \varphi_2 = \varphi_2^* = \frac{1}{2\hat{\sigma}_{\mathrm{e}}}(2\hat{\sigma}_2 - \hat{\sigma}_1) \tag{80}$$

where

$$\hat{\sigma}_{\mathrm{e}} = \sqrt{\hat{\sigma}_1^2 + \hat{\sigma}_2^2 - \hat{\sigma}_1 \hat{\sigma}_2} \tag{81}$$

5.2. Scalar damage law

We choose

$$\Upsilon = \frac{f(\alpha) A^2}{2S} \quad \text{with } f(\alpha) = \frac{1}{(1-\alpha)^m} \tag{82}$$

where S is the damage modulus and m is the damage exponent. The choice $m \neq 1$ represents a trivial, but quite important, generalization of the classical damage law suggested originally by Lemaitre (1971). Upon evaluating the response of a tension bar, which is assumed to deform in a homogeneous fashion in the softening regime and behave perfectly plastic in the absence of damage, we may relate the damage modulus S to the non-dimensional ductility measure s and the exponent m as

$$S = \frac{ms}{2}\sigma_y \varepsilon_y^2, \quad s = \frac{\varepsilon_f}{\varepsilon_y} - 1 \tag{83}$$

where ε_y is the yield strain $(= \sigma_y/E)$ and ε_f is the fracture strain (when $\sigma = 0$). By allowing from $m > 1$, the model can be tuned to mimic real material behavior in quite a realistic fashion.

Remark: Although the damage law in (82) looks the same for the different damage formats, there is a difference w.r.t. the definition of A. •

To complete the model definition, we choose the function $g(x)$ as in (36).

6. Bifurcation results

6.1. Evaluation of bifurcation conditions at the onset of yielding

At the onset of yielding (corresponding to Situation 1 in Section 4.1) there is no damage, i.e. $\alpha = 0$. Moreover, $K = 0$ so that $\bar{H} = H$ and $\sigma_e = \sigma_y$.

To get the complete picture of the bifurcation characteristics, results are given for varying values of the ductility measure s. Lemaitre (1971) has given material data for steel with ductility in the range $s \in (3, 178)$, whereby the lower and upper bounds correspond to very brittle and very ductile damage development, respectively. For any position on the initial yield surface (defined by the angle φ from the biaxial compression state in principal strain space, cf. Fig. 2), we may calculate θ_{cr} and the corresponding $H_{cr,0}$ $(= \bar{H}_{cr,0}$ since $K = 0$ initially). These results are shown in Figs. 2–7 for the three different models (discussed above).

It appears that the result for very ductile damage ($s = 178$) is virtually identical to that of no damage development independently of the damage model used. However, brittle damage ($s = 3$) promotes localization significantly (in the hardening range), and these results are strongly model-dependent.

6.2. Evaluation of bifurcation conditions for prescribed hardening

Let us consider the situation in which $H > H_{cr,0}$ (for a specific strain path), where H is a fixed hardening modulus, and where $H_{cr,0}$ is the critical value of the hardening modulus at the onset of yielding along the specific path under consideration. It is noted that giving H a prescribed value does not infer that $\bar{H}$ is fixed, since $\bar{H}$ will vanish when the saturation $K = K_\infty$ has been reached for very large strain, according to (79). For the present analysis we choose $K_\infty/\sigma_y = 0.3$. Let $(H_{cr,0})_{max}$ denote the maximal value of $H_{cr,0}$ along the entire initial yield surface. For example, for Model 1 (isotropic damage), it appears from Fig. 5 that $(H_{cr,0})_{max}/2G \approx 0.275$ for $s = 3$, while $(H_{cr,0})_{max}/2G \approx 0.00462$ for $s = 178$. Hence, for given $H > (H_{cr,0})_{max}$ development of damage will always be necessary in order to achieve a state where bifurcation can exist. Clearly, this state depends on the actual ductility measure s. Obviously, the analysis necessitates numerical integration of the constitutive relations along the prescribed straight load paths (defined by φ).

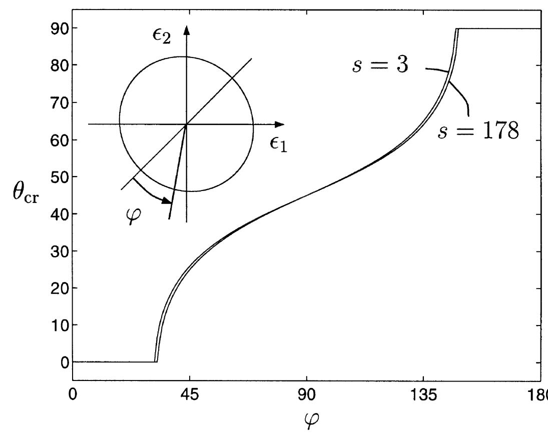

Fig. 2. Variation of θ_{cr} along initial yield surface for different ductility s (isotropic damage: Model 1 or Model 2, 3 with $g_0 = 1$).

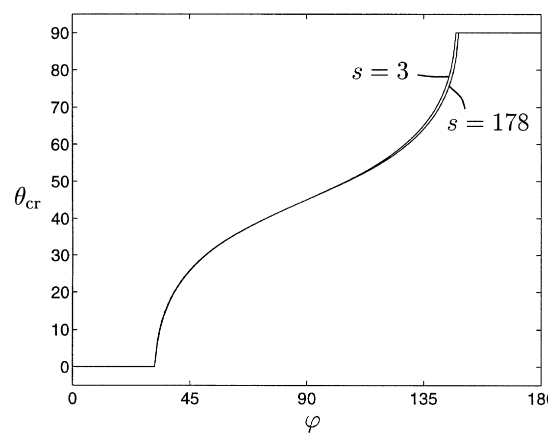

Fig. 3. Variation of θ_{cr} along initial yield surface for different ductility s (Model 2, $g_0 = 0.2$).

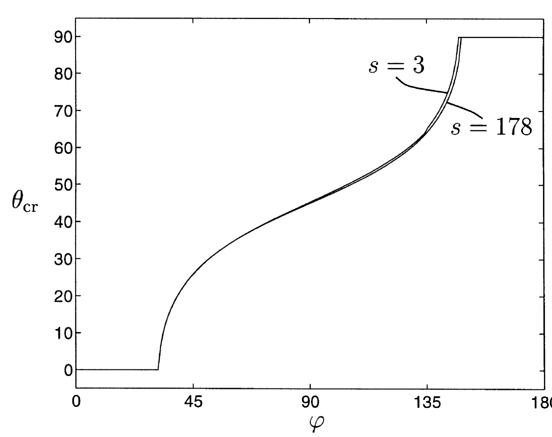

Fig. 4. Variation of θ_{cr} along initial yield surface for different ductility s (Model 3, $g_0 = 0.2$).

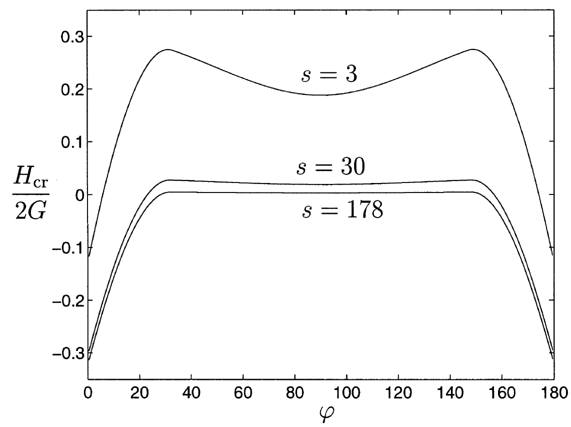

Fig. 5. Variation of $H_{cr,0}$ along initial yield surface for different ductility s (isotropic damage: Model 1 or Model 2, 3 with $g_0 = 1$).

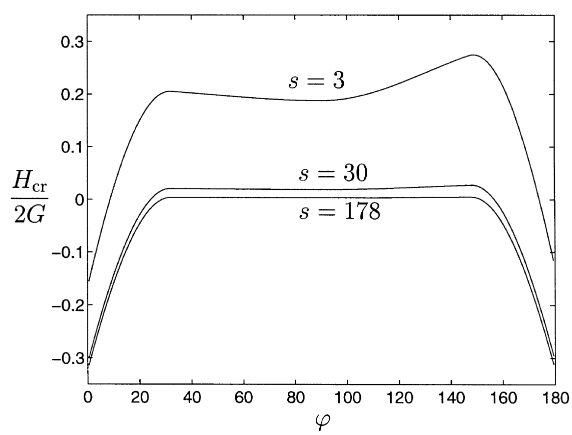

Fig. 6. Variation of $H_{cr,0}$ along initial yield surface for different ductility s (Model 2, $g_0 = 0.2$).

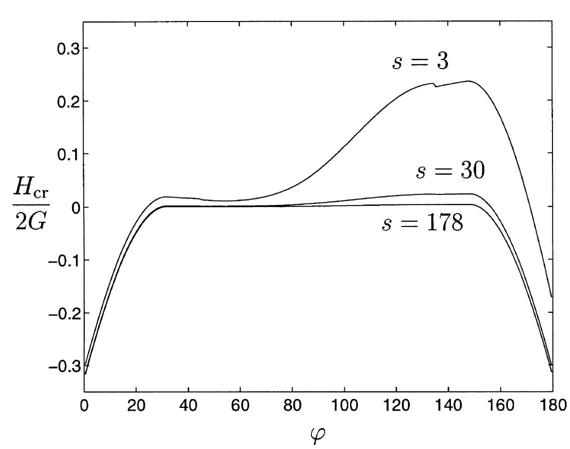

Fig. 7. Variation of $H_{cr,0}$ along initial yield surface for different ductility s (Model, 3, $g_0 = 0.2$).

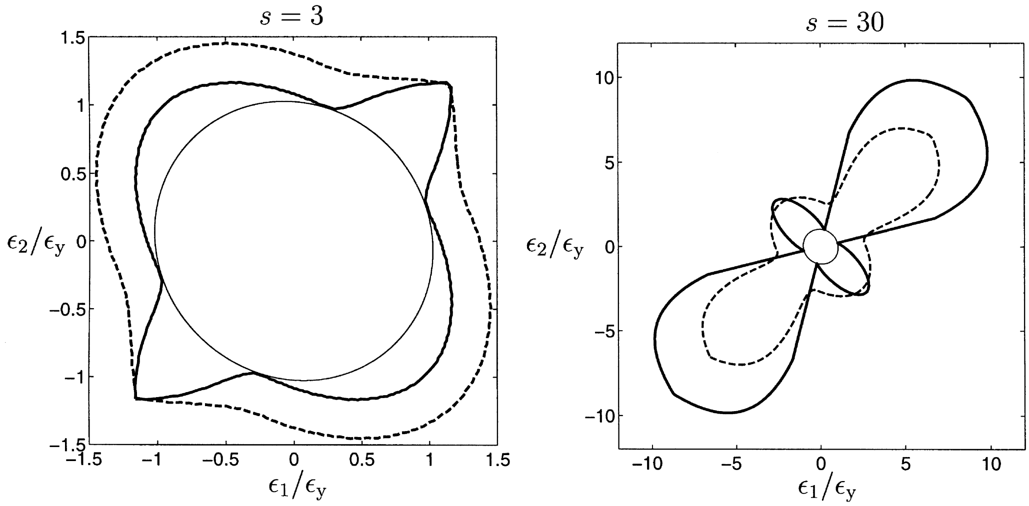

Fig. 8. Bifurcation surfaces for $H = \max(H_{cr,0})$ (—) and $H = 1.2G > \max(H_{cr,0})$ (- -) outside the initial yield surface (–) for different ductility s (isotropic damage: Model 1 or Model, 2, 3 with $g_0 = 1$).

It is possible to construct the 'bifurcation surface' in stress space (similar to the initial yield surface and the saturation surface). However, it is more illuminatg to represent the bifurcation surface in strain space. These surfaces are shown (together with the initial yield surface in strain space) in Figs. 8–10 for the three considered damage models. A brittle response ($s = 3$) and a more ductile response ($s = 30$) are shown. (Although not depicted here due to limited space, results for $s = 30$ are very close to those for very ductile behavior represented by $s = 178$).

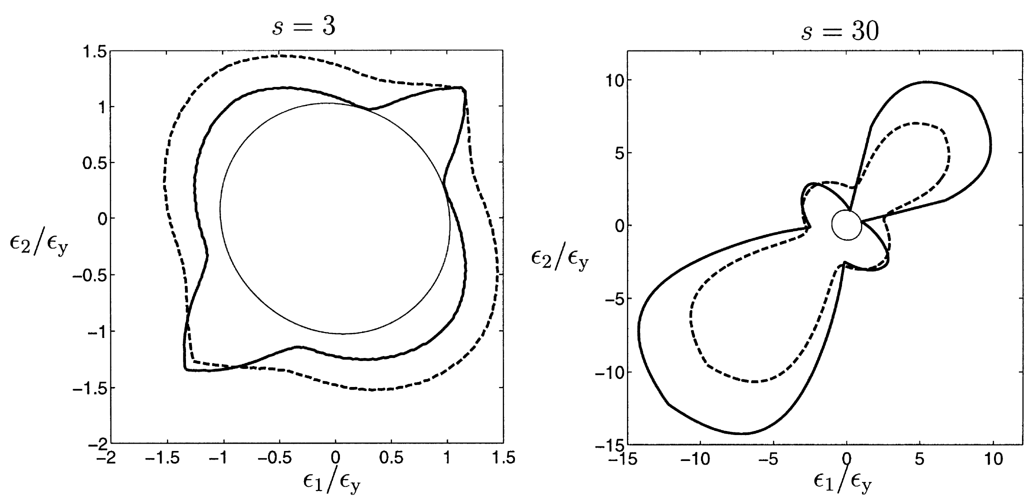

Fig. 9. Bifurcation surfaces for $H = \max(H_{cr,0})$ (—) and $H = 1.2G > \max(H_{cr,0})$ (- -) outside the initial yield surface (–) for different ductility s (Model 2, $g_0 = 0.2$).

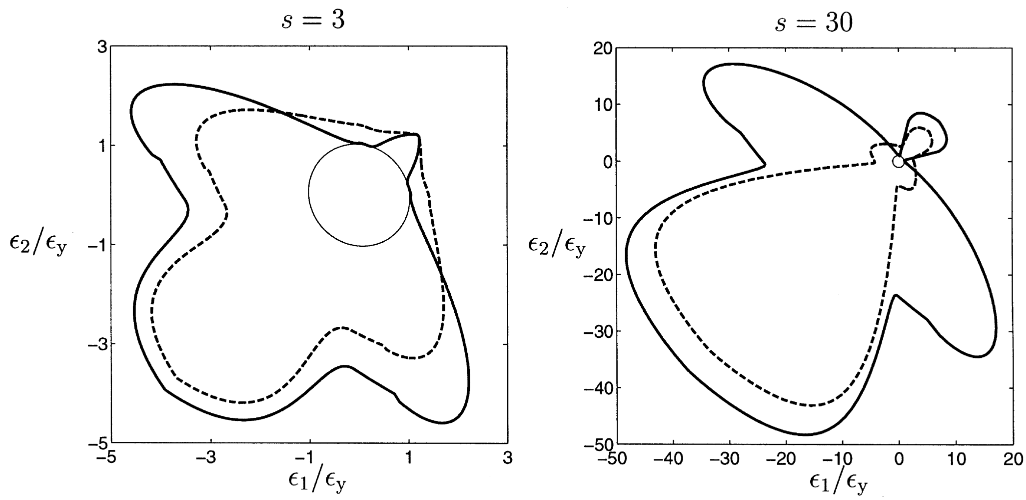

Fig. 10. Bifurcation surfaces for $H = \max(H_{cr,0})$ (—) and $H = 1.2G > \max(H_{cr,0})$ (- -) outside the initial yield surface (–) for different ductility s (Model $g_0 = 0.2$).

Remark: The bifurcation surface is path-independent only in the sense that it is unique for prescribed straight strain paths. •

We may also investigate how the critical value α_{cr} depends on the particular choice of H, and these results are shown in Figs. 11–13. Due to space limitations we omit presentation of the corresponding critical orientation θ_{cr}. Generally, the influence on θ_{cr} of the ductility measure s is somewhat greater than for bifurcation at initial yielding (as shown in Figs. 2–4).

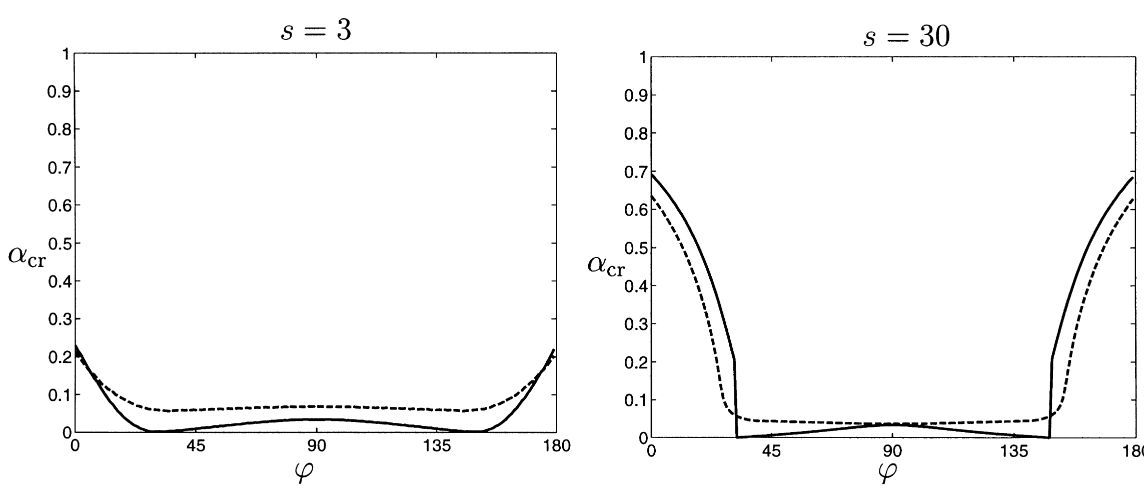

Fig. 11. Variation of α_{cr} along the bifurcation surface for $H = \max(H_{cr,0})$ (—) and $H = 1.2G > \max(H_{cr,0})$ (- -) and for different ductility s (isotropic damage: Model 1 or Model 2,3 with $g_0 = 1$).

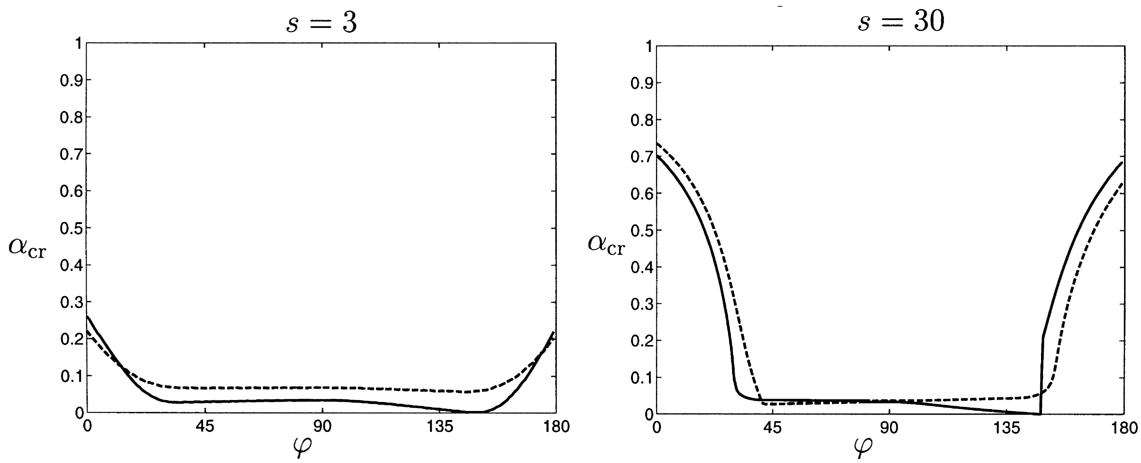

Fig. 12. Variation of α_{cr} along the bifurcation surface for $H = \max(H_{cr,0})$ (—) and $H = 1.2G > \max(H_{cr,0})$ (- -) and for different ductility s (Model 2, $g_0 = 0.2$).

7. Concluding remarks

In this paper we have outlined the thermodynamic framework for (a single scalar) damage that is kinetically coupled to plastic deformation, and the corresponding CTS-tensor is derived. Even when a scalar damage variable is employed, it is possible to go beyond the trivial case of isotopic damage to incorporate MCR-effects. Moreover, it seems quite straightforward to generalize the formulation to tensorial damage, although the actual evaluation (for example, of the CTS-tensor) will be considerably more technical.

The general conditions for discontinuous bifurcations, which define the onset of band-shaped localization, were established along the route laid out by Runesson et al. (1991). It was demonstrated

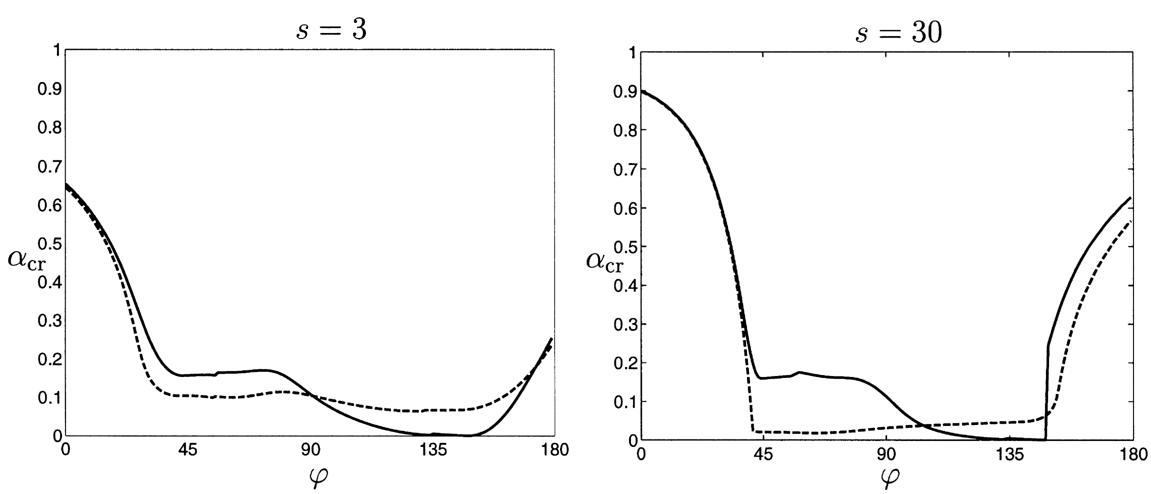

Fig. 13. Variation of α_{cr} along the bifurcation surface for $H = \max(H_{cr,0})$ (—) and $H = 1.2G > \max(H_{cr,0})$ (- -) and for different ductility s (Model 3, $g_0 = 0.2$).

that it is possible to obtain closed-form expressions for the critical band direction and the corresponding state (in terms of $\bar{H}$ and α) under the plane stress condition and in the presence of damage. Numerical results have been presented for a specific model, based on the von Mises criterion with nonlinear (saturation) hardening and a damage law that generalizes that of Lemaitre (1992). The following specific conclusions were reached:

As to the critical angle θ_{cr}, this seems to be quite insensitive to the rate of damaged evolution, i.e., whether the response is ductile or brittle. In fact, the critical orientation virtually coincides with that obtained from pure elastoplasticity (without any damage). However, the investigation was limited to the situation at the onset of yielding.

As to the critical hardening $\bar{H}_{cr}$, this is significantly influenced by the ductility characteristics, such that bifurcation in the hardening regime is obtained for brittle behavior. However, it is confirmed that the results for pure plasticity is approached for very large ductility (low rate of damage development). More specifically, bifurcation will occur at the state of saturation (where $\bar{H} = 0$) in such a case. Moreover, for high rate of damage, the MCR-effect becomes quite pronounced, which is shown manifested in the form of smaller value of $\bar{H}_{cr}$ in the compressive regime. The characteristic behavior is depicted also in Figs. 11–13, which show how the critical amount of damage depends on the ductility after plastic deformation has developed. Clearly, the underlying analysis requires the proper numerical integration of the appropriate evolution equations.

Acknowledgements

We wish to acknowledge support from the Swedish National Research Program in Aeronautics (NFFP), ABB Stal Inc., Saab Military Aircraft Inc., and Volvo Aero Corporation.

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